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OPERATOR ALGEBRAS

On Isometric Representations of the Semigroup $\mathbb{Z}_+\backslash\{1\}$

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Abstract—In this paper we study isometric representations of the semigroup $\mathbb{Z}_+\backslash\{1\}$. The notion of inverse representation is introduced and a complete (to within unitary equivalence) description of such representations of that semigroup is provided. A class of irreducible non-inverse representations (β-representations of the semigroup $\mathbb{Z}_+\backslash\{1\}$) is described.

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1. INTRODUCTION

In the paper [4] Coburn proved that all semiunitary representations of the semigroup of nonnegative integers by isometric operators generate canonically isomorphic algebras. Later a similar result for semigroups with archimedian order and total order have been proved by Douglas [5] and Murphy [7], respectively. In [1] was proved that all non-unitary isometric representations of a semigroup S generate canonically isomorphic C^*-algebras if and only if the natural order on S is total. A simple example of a semigroup with non-total order provides the semigroup $\mathbb{Z}_+\backslash\{1\}$, which originally was discussed by Murphy [7]. Later, Jang [6] pointed out two representations of this semigroup that generate canonically non-isomorphic C^*-algebras. Raeburn and Vittadello [9] studied all isometric representations of the semigroup $\mathbb{Z}_+\backslash\{1\}$ under certain condition.

The present paper is devoted to the isometric representations of the semigroup $\mathbb{Z}_+\backslash\{1\}$. Here the term *isometric representation* stands for a representation by isometric (more precisely, semiunitary) operators in a Hilbert space. We introduce a notion of inverse representation and show that there exist only two inverse irreducible representations (to within unitary equivalence), which are the same representations as in [6], [7], [9], [10]. We also study non-inverse representations of the semigroup $\mathbb{Z}_+\backslash\{1\}$.

2. INVERSE REPRESENTATIONS

Throughout the paper S will stand for an abelian additive cancelative semigroup containing the neutral element 0 and not containing a group different from trivial. By Γ we denote the Grothendieck group generated by the semigroup S. Recall that the group Γ is a quotient of the semigroup $S \times S$ with respect to equivalence $(a, b) \sim (c, d)$ if and only if $a + d = b + c$, and the inverse of the quotient class $[(a, b)]$ is $[(b, a)]$. The notation $\Gamma = S - S$ is commonly used.

Let $T : S \to B(H_T)$ be the *faithful non-unitary isometric* representation of the semigroup S into algebra $B(H_T)$ of all bounded linear operators on the Hilbert space H_T. Observe that in this case $T(0) = I$.

For any $a \in S$ by $T^*(a)$ we denote the conjugate of the operator $T(a)$. We have $T^*(a)T(a) = I$, where I is the identity operator, and $T(a)T^*(a) = P_T(a)$, where $P_T(a)$ is the projection ($P_T(a) \neq I$).

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Let $T : S \to B(H_T)$ be an isometric representation of the semigroup S. An element $h_0 \in H_T$ we call *initial* for the operator $T(a)$ if $T^*(a)h_0 = 0$ for any $a \in S\backslash\{0\}$ and $||h_0|| = 1$.

The operators $T(a)$ and $T^*(b)$, $a, b \in S$, we call *trivial monomials*. A *monomial* is defined to be a finite product of trivial monomials. The set of all monomials forms a multiplicative involutive semigroup, which we will denote by S_T^*. On the semigroup S we define the order: $a \prec b$ if $b = a + c$. Note that S is a net with respect to this order.

Lemma 2.1. *For any monomial V there exist x and y in S such that*

$$\lim_{c \in S} T^*(c)VT(c) = T^*(x)T(y),$$

where $\lim_{c\in S}$ is the limit by the net S.

Proof. Let V be a monomial represented as follows

$$V = \prod_{i=1}^{n} T^{'}(a_i),$$

where $T^{'}(a_i)$ is either $T(a_i)$ or $T^*(a_i)$. We split the set $\{a_i\}_{i=1}^n$ into two subsets: $\{a_{i_k}\}_{k=1}^l$ and $\{a_{i_j}\}_{j=l+1}^n$. The first subset consists of those $\{a_i\}_{i=1}^n$ for which the monomial V involves the operator $T(a_i)$, while for the second V involves the operator $T^*(a_i)$. Let

$$a = \sum_{j=l+1}^{n} a_{i_j},\ b = \sum_{k=1}^{l} a_{i_k}.$$

Using the equalities $T^*(s)T(s) = I$ and $T(s)T(t) = T(t)T(s)$ for any $s, t \in S$ we have

$$T^*(c)VT(c) = T^*(a)T(b),$$

where $c = a + b$. Therefore for any d, $c \prec d$

$$T^*(d)VT(d) = T^*(a)T(b) = \lim_{c\in S} T^*(c)VT(c),$$

and the result follows.

Observe that if $T^*(a)T(b) = T^*(c)T(d)$ for some a, b, c and d from S, then due to faithfulness of the representation we have $b + c = a + d$. This implies that to each monomial V can be correspond a unique element $b - a$ from the Grothendieck group Γ. The element $b - a$ we call an *index* of the monomial V and denote by $\mathrm{ind}(V) = b - a$. This notion was introduced in [2] for regular representation of semigroup S.

Lemma 2.2. *The following assertions hold:*

1. *The index of a monomial does not depend on its representation by elementary monomials.*
2. *The index of a product of monomials is equal to the sum of indices of factors:*

$$\mathrm{ind}(V_1 \cdot V_2) = \mathrm{ind}(V_1) + \mathrm{ind}(V_2).$$

Denote by $S_{0,T}^*$ the subsemigroup of the semigroup S_T^* consisting of those V for which $\mathrm{ind}(V) = 0$.

An isometric representation $T : S \to B(H_T)$ is called *inverse representation* if S_T^* is an inverse semigroup with respect to multiplication and involution, or equivalently, if $S_{0,T}^*$ is a semigroup of idempotents in S_T^* (i.e., a semigroup of orthogonal projections). According to lemma 2.2 from [2] each semigroup S possesses at least one inverse representation. On the other hand, as it was shown in [1], if the above defined order on S is a total order, then all the isometric representations of S are inverse.

A simple example of inverse representation is the representation $L : \mathbb{Z}_+ \to B(l^2(\mathbb{Z}_+))$ by the shift operator $L(n)e_m = e_{n+m}$, where $e_n(m) = \delta_{n,m}$ is the Kronecker symbol. Notice that the system $\{e_n(m)\}$ forms an orthonormal basis in $l^2(\mathbb{Z}_+)$, and in this case, the semigroup ${\mathbb{Z}_+}_L^*$ is a bicyclic semigroup.

3. INVERESE REPRESENTATIONS OF THE SEMIGROUP $\mathbb{Z}_+\backslash\{1\}$

Below we use the notation $M = \mathbb{Z}_+\backslash\{1\}$. Murphy [7], and then Jang [6], have proved that there exist at least two isometric representations of the semigroup M that generate canonically non-isomorphic C^*-algebras. In [9] were studied isometric representations T of the semigroup M, when the projections $T(n)T^*(n)$ and $T(m)T^*(m)$ commute for any $n, m \in M$. It follows from the results of this section that T is an inverse representation. Here, in fact, we obtain the same results as in [9], but now their proofs are based on the notion of *initial element*. This section is included for completeness of presentation of the topic.

Consider two isometric representations of the semigroup M:

$$T_1 : M \to B(l^2(M)), \quad T_0 : M \to B(l^2(\mathbb{Z}_+)),$$

determined by the shift operators $T_1(m)e_n = e_{m+n}$ and $T_0(m)f_n = f_{m+n}$, where $\{e_n\}_{n\in M}$ and $\{f_n\}_{n\in\mathbb{Z}_+}$ are the natural orthonormal bases in $l^2(M)$ and $l^2(\mathbb{Z}_+)$, respectively. In [7] and [6] it was shown that these representations are not unitary equivalent.

The representations T_0 and T_1 generate inverse semigroups $M^*_{T_0}$ and $M^*_{T_1}$. The representation T_1 is inverse because it is regular (see [1], [2]), while T_0 is inverse because $T_0^*(2)T_0(3)$ is a shift operator on $l^2(\mathbb{Z}_+)$ transferring the basis elements e_n into e_{n+1}. It is clear that $(T_0^*(2)T_0(3))^n = T_0(n)$, where $n \geq 2$. Observe that $M^*_{T_0}$ is a bicyclic semigroup with a generator $T_0^*(2)T_0(3)$.

Below we show that each inverse faithful isometric irreducible representation of the semigroup M is unitary equivalent to either T_0 or T_1. Let T be an irreducible isometric representation of semigroup M.

Lemma 3.1. *For any $n \geq 2$ the equality holds:*

$$T^*(n)T(n+1) = T^*(n+1)T(n+2).$$

Proof. It follows from the equality $T^*(m)T(m) = I$ that

$$T^*(n)T(n+1) = T^*(n)T^*(3)T(3)T(n+1) = T^*(n+3)T(n+4)$$

$$= T^*(n+1)T^*(2)T(2)T(n+2) = T^*(n+1)T(n+2),$$

yielding the result.

Corollary 3.1. *For all $l, n, m \in M$ we have*

$$T^*(n)T(m) = T^*(n+l)T(m+l).$$

Below we use the following elementary identities:

$$T^*(2) = T^*(3)T^*(2)T(3),\ T^*(3) = T^*(4)T^*(2)T(3),\ T^*(3)T(2) = T^*(2)T^*(2)T(3),$$

which imply: if $h_0 \in \ker T^*(2)T(3)$, then

$$T^*(2)T(3)h_0 = T^*(2)h_0 = T^*(3)h_0 = T^*(3)T(2)h_0 = 0.$$

Lemma 3.2. *Let $T : M \to B(H)$ be an irreducible representation, and let* $\ker T^*(2)T(3) \neq 0$. *Then T is inverse and is unitary equivalent to the representation T_1.*

Proof. We fix $h_0 \in \ker T^*(2)T(3)$ to satisfy $||h_0|| = 1$. Denote by H_0 the Hilbert subspace in H, generated by the linear combinations of the elements of the set $\{h_n\}_{n\in M}$, where $h_n = T(n)h_0$. It is clear that $||h_n|| = 1$.

We show that $H_0 = H$. Observe first that since $T(n)H_0 \subset H_0$ it is enough to show that $T^*(n)h_m$ belongs to H_0 for all $n \in M$. The above equalities imply

$$T^*(2)h_0 = 0, \quad T^*(2)h_3 = T^*(2)T(3)h_0 = 0,$$

$$T^*(3)h_0 = 0, \quad T^*(3)h_2 = T^*(2)T^*(2)T(3)h_0 = 0,$$

$$T^*(3)h_4 = T^*(3)T(4)h_0 = T^*(2)T(3)h_0 = 0.$$

Now, taking into account that if $n \in M\backslash\{0;3\}$, then $n-2$ also belongs to M, we obtain

$$T^*(2)h_n = T^*(2)T(2)h_{n-2} = h_{n-2}.$$

Similarly $T^*(3)h_m = h_{m-3}$ for $m \in M\backslash\{0,2,4\}$.

Let now m be an arbitrary number from M. Then $m = 3l + 2k$, $T^*(m) = T^*(3)^l T^*(2)^k$. Therefore either $T^*(m)h_n = 0$ or $T^*(m)h_n = h_{n-m}$, implying that $H_0 = H$.

Next, we show that the set $\{h_n\}_{n\in M}$ forms an orthonormal basis in H_0. Let $n, m \in M$ and $n > m$. Assuming that $n - m \neq 1$, we have

$$(h_m, h_n) = (T^*(n-m)h_0, h_0) = 0.$$

If $n = m+1$, then $T^*(m)T(n) = T^*(2)T(3)$ and $(h_m, h_n) = (h_0, T^*(2)T(3)h_0) = 0$. Finally, we introduce the operator $U : H_0 \to l^2(M)$, $Uh_n = e_n$, and observe that $U^*T_1U = T$. This completes the proof.

Lemma 3.3. *Let $T : M \to B(H)$ be an inverse representation. Then*

$$P = T^*(3)T(2)T^*(2)T(3) \text{ and } Q = T^*(2)T(3)T^*(3)T(2)$$

are projections and $Q < P$.

Proof. It is easy to check that P and Q are projections. So, we prove that $Q < P$. Using Lemma 3.1 and the equalities

$$T(2)T^*(2)T(3)T^*(3) = T(3)T^*(3)T(2)T^*(2), \quad T^*(2)T(4) = T(2),$$

we can write

$$PQ = T^*(3)T(2)T^*(2)T(3)T^*(2)T(3)T^*(3)T(2)$$

$$= T^*(3)T(2)T^*(2)T(3)T^*(3)T(4)T^*(3)T(2)$$

$$= T^*(3)T(3)T^*(3)T(2)T^*(2)T(4)T^*(3)T(2) = T^*(3)T(2)T^*(2)T(2)T(2)T^*(3)T(2)$$

$$= T^*(3)T(2)T(2)T^*(3)T(2) = T^*(3)T(4)T^*(3)T(2) = T^*(2)T(3)T^*(3)T(2) = Q,$$

and the result follows.

Lemma 3.4. *Let T be a non-unitary isometric representation of the semigroup M. Then there exists an element which is initial for any operator $T(n)$.*

Proof. Denote $H_2 = \ker T^*(2)$. It is clear that $T^*(3) : H_2 \to H_2$. Indeed,

$$T^*(2)T^*(3)H_2 = T^*(3)T^*(2)H_2 = 0.$$

Thus, two cases are possible: either $T^*(3)H_2 = \{0\}$ or $T^*(3)H_2 \neq \{0\}$. In the first case as an initial element can be taken any element from H_2. As for the second case, we fix a non-zero element $h_2 \in H_2$, then $h_0 = T^*(3)h_2||T^*(3)h_2||^{-1}$ will be an initial element because we can apply to h_0 the operator $T^*(3)$ and $T^*(6) = T^*(2)T^*(2)T^*(2)$. The proof is complete.

Lemma 3.5. *Let $T : M \to B(H)$ be an irreducible non-unitary inverse representation of M, and let* $\ker T^*(2)T(3) = \{0\}$. *Then T is unitary equivalent to the representation T_0.*

Proof. It follows from the condition $\ker T^*(2)T(3) = \{0\}$ that $P = I$ is an identity operator and $T^*(2)T(3)$ is an isometry. We show that $T^*(2)T(3)$ is a non-unitary operator. Indeed, according to Lemma 3.1 we have

$$T^*(2)T(3)T^*(2)T(3) = T^*(2)T(3)T^*(3)T(4) = T^*(2)T(3)T^*(3)T(2)T(2) = QT(2).$$

The operator $QT(2)$ is non-unitary because it is a product of a projection and an isometric operator. Hence $T^*(2)T(3)$ is also a non-unitary isometric operator. By the Wold-von Neumann decomposition theorem (see, e.g., [8]) the isometric operator $T^*(2)T(3)$ can be represented as a direct sum of a shift operator and a unitary operator. Let $h_0 \in H$ be an initial element for the operator $T^*(2)T(3)$.

We set $h_n = (T^*(2)T(3))^n h_0$. Since $P = I$, we have $||h_n|| = 1$ and $(h_n, h_m) = 0$ for $n \neq m$, that is, $\{h_n\}_{n=0}^{\infty}$ is an orthonormal system in H. It follows from the equalities $1 = ||h_2|| = ||QT(2)h_0||$ and $||T(2)h_0|| = 1$ that $QT(2)h_0 = T(2)h_0$, that is, $h_2 = T(2)h_0$. Similarly we obtain

$$h_3 = (T^*(2)T(3))T(2)h_0 = T(3)h_0.$$

Continuing this process we get $h_n = T(n)h_0$. Thus, a Hilbert subspace H_0 of H for which the system $\{h_n\}_{n=0}^{\infty}$ forms an orthonormal basis is invariant for the shift operator $T^*(2)T(3)$ and on H_0 we have $(T^*(2)T(3))^n = T(n)$, $n \in M$. Hence $H_0 = H$ and the unitary operator

$$U : H_0 \to l^2(\mathbb{Z}_+), \quad Uh_n = e_n,$$

is an intertwining operator, that is, $U^*T_0U = T$. This completes the proof.

From Lemmas 3.2 and 3.5 we obtain the following results.

Theorem 3.1. *Let T be a non-unitary inverse representation of the semigroup M. Then T is unitary equivalent to either T_0 or T_1.*

Theorem 3.2. *Each isometric inverse representation T of the semigroup M can be represented as a direct sum*

$$T = kT_0 \oplus lT_1 \oplus T_2,$$

where k and l are the multiplicities of the representations T_0 and T_1, respectively, and T_2 is a unitary representation of the semigroup M.

Remark 3.1. It can be shown that for any isometric representation T of the semigroup M the inverseness of the semigroup M_T^* is equivalent to the commutativity of *elementary projections* $P_T(n) = T(n)T^*(n)$, $n \in M$. The following question is of interest: do the notions of inverseness and permutability of the elementary projections for an isometric representation of a general semigroup equivalent?

4. β-REPRESENTATION OF THE SEMIGROUP M

It follows from Coburn's theorem [4] that there is only one (to within unitary equivalence) infinite irreducible representation of the semigroup $\mathbb{Z}_+$. In this section we prove that for the "deformed" semigroup M the number of such representations is continuum.

Let H_0 be a Hilbert subspace of the space $l^2(\mathbb{Z}_+)$ generated by the basis $\{e_n\}_{n=2}^{\infty}$, $e_n(m) = \delta_{n,m}$. Denote by H_β the Hilbert subspace of $l^2(\mathbb{Z}_+)$ generated by the elements from H_0 and the element $g_\beta = \beta e_0 + te_1$, where $\beta \in \mathbb{C}, t \in \mathbb{R}_+$ and $\beta^2 + t^2 = 1$. It is clear that the family $\{g_\beta, \{e_n\}_{n=2}^{\infty}\}$ forms an orthonormal basis in H_β and $H_\beta = \mathbb{C}g_\beta \oplus H_0$.

Let $P_\beta : l^2(\mathbb{Z}_+) \to H_\beta$ be the orthogonal projection from $l^2(\mathbb{Z}_+)$ onto H_β. Define the representation $\tau_\beta : M \to B(H_\beta)$: $\tau_\beta(n) = P_\beta T_0(n) P_\beta$, where $T_0 : M \to B(l^2(\mathbb{Z}_+))$ is the inverse representation defined above. Since $T_0(n)$ maps H_β onto itself, the representation $\tau_\beta : M \to B(H_\beta)$ is an isometric representation of the semigroup M.

Lemma 4.1. *The representation $\tau_\beta : M \to B(H_\beta)$ is an inverse representation if and only if either $\beta = 0$ or $|\beta| = 1$.*

Proof. If $\beta = 0$, then $g_0 = e_1$ and $H_0 = l^2(\mathbb{N})$. Hence the representation $\tau_0 : M \to B(l^2(\mathbb{N})$ is unitary equivalent to representation T_0.

If $|\beta| = 1$, then $g_\beta = \beta e_0$ and $H_\beta = l^2(M)$, and in this case the representation $\tau_\beta : M \to B(l^2(M))$ is unitary equivalent to representation T_1.

Now assuming $0 < |\beta| < 1$, we show that the operator $\tau_\beta^*(3)\tau_\beta(2)\tau_\beta^*(2)\tau_\beta(3)$ is not a projection. We first evaluate $\tau_\beta^*(2)\tau_\beta(3)$ on basis elements g_β and e_n of the space H_β. We have $\tau_\beta^*(2)\tau_\beta(3)e_n = e_{n+1}$, $n = 2, 3, \ldots$, that is, $\tau_\beta^*(2)\tau_\beta(3)$ is an one-sided shift operator on H_0. Hence the contraction of $\tau_\beta^*(3)\tau_\beta(2)\tau_\beta^*(2)\tau_\beta(3)$ on H_0 is an identity operator. We have

$$(\tau_\beta^*(2)\tau_\beta(3)g_\beta, g_\beta) = (\tau_\beta(3)g_\beta, \tau_\beta(2)g_\beta) = \beta t,$$

$$(\tau^*_\beta(2)\tau_\beta(3)g_\beta, e_2) = (\tau_\beta(3)g_\beta, e_4) = t, \quad (\tau^*_\beta(2)\tau_\beta(3)g_\beta, e_n) = 0 \quad \text{for } n \geq 3.$$

Hence $\tau^*_\beta(2)\tau_\beta(3)g_\beta = \beta t g_\beta + te_2$, implying

$$\tau_\beta(2)(\beta t g_\beta + te_2) = \beta t(\beta e_2 + te_3) + te_4.$$

Now we evaluate $\tau^*_\beta(3)e_2$, $\tau^*_\beta(3)e_3$ and $\tau^*_\beta(3)e_4$. It is clear that

$$(\tau^*_\beta(3)e_2, g_\beta) = 0 = (\tau^*_\beta(3)e_2, e_n), \ n \geq 2,$$

implying $\tau^*_\beta(3)e_2 = 0$. Next, we have

$$(\tau^*_\beta(3)e_3, e_n) = (T^*(3)e_4, e_n) = 0 \text{ for all } n = 2, 3, \ldots, \text{ and}$$

$$(\tau^*_\beta(3)e_3, g_\beta) = (e_3, \beta e_3 + te_4) = \bar{\beta}, \ (\tau^*_\beta(3)e_4, g_\beta) = (e_4, \beta e_3 + te_4) = t.$$

Hence $\tau^*_\beta(3)e_3 = \bar{\beta} g_\beta$, $\tau^*_\beta(3)e_4 = tg_\beta$. Thus, we have

$$\tau^*_\beta(3)\tau_\beta(2)\tau^*_\beta(2)\tau_\beta(3)g_\beta = (|\beta|^2 t^2 + t^2)g_\beta.$$

It follows from $0 < |\beta| < 1$ and $|\beta|^2 + t^2 = 1$ that $|\beta|^2 t^2 + t^2 < 1$. Hence the operator $\tau^*_\beta(3)\tau_\beta(2)\tau^*_\beta(2)\tau_\beta(3)$ is a diagonal operator with respect to basis $g_\beta, e_2, e_3, \ldots$ with eigenvalues $|\beta|^2 t^2 + t^2$ and 1. Since this operator is not a projection, we conclude that $M^*_{\tau_\beta}$ is not an inverse semigroup. The proof is completed.

Theorem 4.1. *The following assertions hold:*

1. *The representation τ_β of the semigroup M is irreducible and isometric.*
2. *If $\beta_1 \neq \beta_2$ and $|\beta_1| < 1$, then the representations T_{β_1} and T_{β_2} are unitary equivalent.*

Proof. (1) Observe first that if $|\beta|^2 t^2 + t^2 = 1$, then $\beta = 0$. Hence in this case we obtain a representation τ_0 which, as it was mentioned above, is not only irreducible and isometric but also inverse. For $|\beta|^2 t^2 + t^2 \neq 1$ we set

$$Q_\beta = c(I - \tau^*_\beta(3)\tau_\beta(2)\tau^*_\beta(2)\tau_\beta(3)), \text{ where } c = \frac{1}{1 - |\beta|^2 t^2 - t^2},$$

and observe that Q_β is a projection onto the one-dimensional space $\mathbb{C}g_\beta$, while
$A_\beta = (I - Q_\beta)\tau^*_\beta(2)\tau_\beta(3)(I - Q_\beta)$ is an one-sided shift operator on H_0 which vanishes on g_β, that is, $A_\beta g_\beta = 0$, $A_\beta e_n = e_{n+1}$ for $n = 2, 3, \ldots$. Therefore the C^*-algebra generated by the operator A_β is a Toeplitz algebra on H_0 containing all the compact operators from $(1 - Q_\beta)B(H_\beta)(1 - Q_\beta)$. Therefore each invariant subspace of C^*-algebra generated by the representation τ_β either contains H_0 or is orthogonal to H_0. To complete the proof of assertion (1), it remains to observe that since $H_\beta = \mathbb{C}g_\beta \oplus H_0$ and $\tau_\beta(2)g_\beta \in H_0$, the representation τ is irreducible.
(2) Let $\tau_{\beta_1} \sim \tau_{\beta_2}$ and $0 < |\beta_1| < 1$. We show that $\beta_1 = \beta_2$. Let $U : H_{\beta_1} \to H_{\beta_2}$ be a unitary operator such that $\tau_{\beta_1} = UT_{\beta_2}U^*$. Since each element from H_{β_i} on which the operators $\tau^*_{\beta_i}(2)$ and $\tau^*_{\beta_i}(3)$ vanish is a multiple of the element g_{β_i}, $i = 1, 2$, the unitary operator $U : H_{\beta_2} \to H_{\beta_2}$ transfers the space $\mathbb{C}g_{\beta_1}$ on $\mathbb{C}g_{\beta_2}$, that is, $Ug_{\beta_1} = e^{i\theta}g_{\beta_2}$, $0 \leq \theta \leq 2\pi$. Next, since $V = e^{-i\theta}U$ is a unitary operator, we have $\tau_{\beta_1} = V\tau_{\beta_2}V^*$ and $Vg_{\beta_1} = g_{\beta_2}$. Therefore

$$\beta_1 t_1 = (\tau^*_{\beta_1}(2)\tau_{\beta_1}(3)g_{\beta_1}, g_{\beta_1}) = (\tau^*_{\beta_2}(2)\tau_{\beta_2}(3)g_{\beta_2}, g_{\beta_2}) = \beta_2 t_1,$$

and $|\beta_1|^2 t_1^2 + t_1^2 = |\beta_2|^2 t_2^2 + t_2^2$ are the eigenvalues of the operator $\tau^*_{\beta_i}(3)\tau_{\beta_i}(2)\tau^*_{\beta_i}(2)\tau_{\beta_i}(3)$ on the vector g_{β_i}, $i = 1, 2$. This implies $\beta_1 = \beta_2$. The proof is complete.

The following questions that arise naturally are of interest. To describe the class of all irreducible isometric representations of the semigroup M. Do all the irreducible isometric infinite representations of the semigroup M unitary equivalent to some representation $\tau_\beta : M \to B(H_\beta)$?

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